CONTRIBUTIONS TO THE THEORY OF POLYNOMIALS

by

QAZI MOHAMMAD TARIQ

DEPARTMENT OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY KANPUR

MAY, 1986

CONTRIBUTIONS TO THE THEORY OF POLYNOMIALS

A Thesis Submitted
in Partial Fulfilment of the Requirements
for the Degree of

DOCTOR OF PHILOSOPHY

by

QAZI MOHAMMAD TARIQ

to the

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY KANPUR
MAY, 1986

MATH- 1986- D-TAR-CON

CERTIFICATE

This is to certify that the work embodied in the thesis entitled " CONTRIBUTIONS TO THE THEORY OF POLYNOMIALS" being submitted by Mr. Qazi Mohammad Tariq has been carried out under our supervision and that it has not been submitted for the award of any degree or diploma elsewhere.

May 1986

(O. P. Juneja)   (G. P. Kapoor)

Department of Mathematics
Indian Institute of Technology
Kanpur, INDIA.

ACKNOWLEDGEMENTS

TABLE OF CONTENTS

SYNOPSIS

of the

Ph.D. Dissertation

on

CONTRIBUTIONS TO THE THEORY OF POLYNOMIALS

by

QAZI MOHAMMAD TARIQ

Department of Mathematics
Indian Institute of Technology, Kanpur, India

MAY. 1986

The thesis consists of two main chapters, namely Chapters 1 and 2. In Chapter 1 we consider some problems related to the famous inequalities of Markov and Bernstein about the derivatives of polynomials and trigonometric polynomials. In Chapter 2 we attempt to answer a question inspired by the well-known conjecture of Bl. Sendov.

We use the notation $D(a;R)$ for the open disk $\left\{z \in \mathbb{C} : |z-a| < R\right\}$ and $\overline{D(a;R)}$ for its closure.

The main results of Chapter 1 are:

THEOREM 1.1. <u>Let</u> $f(z) := z\,\dfrac{p(z)}{q(z)}$ <u>where</u> p <u>and</u> q <u>are polynomials of degree</u> μ <u>and</u> ν <u>respectively. If both</u> p <u>and</u> q <u>do not vanish in</u> $D(0;1)$ <u>and</u> $|f(x)| \leq 1$ <u>for</u> $-1 < x < 1$, <u>then</u>

$$|f'(0)| \leq \min_{0 < x \leq 1} \frac{1}{x}\,\frac{(1+x^2)^{\nu/2}}{(1-x^2)^{\mu/2}} .$$

COROLLARY 1.1. Let $p(z) = z\,q(z)$ be an odd polynomial of degree $\leq n$ such that $|p(x)| \leq 1$ for $-1 \leq x \leq 1$. If $q(z) \neq 0$ in $D(0;1)$, then for all $x \in [-1,1]$

$$|p(x)/x| < \sqrt{n}/c_0$$

where c_0 is the only positive root of the equation

$$c\, e^{(1+c^2)/2} = 1.$$

In the absence of any restriction on the zeros of q it was proved by R. P. Boas, Jr. that if T_m is the Chebyshev polynomial of the first kind of degree $m = n$ or $n-1$ according as n is odd or even, then

$$|p(x)| \leq |T_m(x)| \quad (|x| \leq \sin \tfrac{\pi}{2m}).$$

THEOREM 1.2. Let $f(z) := \sum\limits_{\nu=1}^{\infty} a_\nu z^\nu$ be typically real in $D(0;1)$. If $|f(re^{i\alpha})| \leq 1$ for $0 < r < 1$ and some $\alpha \in [0,\pi]$, then

$$|a_1| \leq \lambda_\alpha := \inf_{0<r<1} \frac{1+r}{r(1-r)} \sqrt{1+r^4-2r^2 \cos 2\alpha}\,.$$

The estimate is sharp for each α.

THEOREM 1.3. Let $f(z) := \sum\limits_{\nu=1}^{\infty} a_\nu z^\nu$ be typically real in $D(0,1)$. If $|f(re^{i\alpha})| \leq 1$ for $-1 < r < 1$ and some $\alpha \in [0,\pi)$, then

$$|a_1| \leq \mu_\alpha := \inf_{0<r<1} \frac{1+r^2}{r(1-r^2)} \sqrt{1+r^4-2r^2 \cos 2\alpha}\,.$$

The case $\alpha = 0$ of Theorem 1.3 was first proved by Q. I. Rahman and St. Ruscheweyh by a method different from ours.

THEOREM 1.4. If $p(x) := \sum\limits_{\nu=0}^{n} a_\nu x^\nu$ is a polynomial of degree n such that $p(1) = 0$, then

$$|a_n| \leq \frac{n}{n+1} \frac{(2n)!}{2^n (n!)^2} \left(\frac{2n+1}{2}\right)^{1/2} \left(\int_{-1}^{1} |p(x)|^2\, dx\right)^{1/2}.$$

The inequality is sharp and equality holds for

$$p(x) := P_n(x) - \frac{1}{n^2} \sum_{\nu=0}^{n-1} (2\nu+1) P_\nu(x)$$

where P_ν is the Legendre polynomial of degree ν with the normalization $P_\nu(1) = 1$. Besides,

$$|a_{n-1}| \leq \frac{(n^2+2)^{1/2}}{n+1} \frac{(2n-2)!}{2^{n-1}((n-1)!)^2} (\frac{2n-1}{2})^{1/2} (\int_{-1}^{1} |p(x)|^2 dx)^{1/2}$$

which is again sharp as the example

$$p(x) := \frac{2n+1}{n^2+2} P_n(x) - P_{n-1}(x) + \frac{1}{n^2+2} \sum_{\nu=0}^{n-2} (2\nu+1) P_\nu(x)$$

shows.

THEOREM 1.5. Let $f(z)$ be an entire function of exponential type τ such that $|f(x)| \leq 1$ for $x \in \mathbb{R}$. If $|f(0)| = \cos a$, where $0 \leq a \leq \frac{\pi}{2}$, and $f'(0) = 0$, then

$$|f(x)| \leq \sin(\sqrt{(\pi-a)^2 + \tau^2 x^2} - \frac{\pi}{2}) \text{ for } |x| \leq \frac{\sqrt{a(2\pi-a)}}{\tau}.$$

COROLLARY 1.3. Under the conditions of Theorem 1.5 we have

$$|f''(0)| \leq \frac{\sin a}{a} \tau^2.$$

Besides, we present an alternative proof of the following result of C. Frappier:

THEOREM A'. Let $f(z)$ be an entire function of exponential type τ such that $|f(x)| \leq 1$ for $x \in \mathbb{R}$ and $f(0) = 0$. Then

$$|f(x)| \leq |\sin \tau x| \text{ for } |x| \leq \frac{\pi}{2\tau}.$$

In Chapter 2 we prove:

THEOREM 2.1. Let $|a| = 1$. If $p(z) := c(z-a)^k \prod_{\nu=1}^{n-k} (z-z_\nu)$ is a polynomial of degree n ($> k$) such that $|z_\nu| \leq 1$ for $\nu = 1, \ldots, n-k$, then taking multiplicity into account, $p'(z)$ has at least k zeros in $\overline{D(\frac{a}{k+1}; \frac{k}{k+1})}$ $(\subset \overline{D(a; \frac{2k}{k+1})})$.

THEOREM 2.2. Let $|a| \leq 1$. If $p(z) := c(z-a)^k \prod_{\nu=1}^{n-k} (z-z_\nu)$ is a polynomial of degree n with $k < n \leq (k+1)^2$ such that

$|z_\nu| \leq 1$ <u>for</u> $\nu = 1, \ldots, n-k$, <u>then taking multiplicity into account</u>, $p'(z)$ <u>has at least</u> k <u>zeros in</u> $\overline{D(a; \frac{2k}{k+1})}$.

CERTAIN EXTREMAL PROPERTIES OF POLYNOMIALS, TRIGONOMETRIC POLYNOMIALS AND ENTIRE FUNCTIONS OF EXPONENTIAL TYPE

1.1. E s t i m a t e s   f o r   t h e   d e r i v a t i v e

Let $\mathcal{J}_n$ denote the class of all trigonometric polynomials $t(\theta) := \sum\limits_{\nu=-n}^{n} a_\nu e^{i\nu\theta}$ of degree at most n. The problem of estimating $\left|t'(\theta_0)\right|$ at an arbitrary point $\theta_0 \in \mathbb{R}$ if $t \in \mathcal{J}_n$ and $\left|t(\theta)\right| \leq 1$ for $\theta \in \mathbb{R}$ was first considered by S. Bernstein. It is clearly enough to estimate $\left|t'(0)\right|$. To see this note that if t is a trigonometric polynomial of degree n such that $\left|t(\theta)\right| \leq 1$ for $\theta \in \mathbb{R}$ then the same is true about

$$t_1 : \theta \mapsto t(\theta + \theta_0);$$

besides, $t_1'(0) = t'(\theta_0)$. Further, there is no loss of generality in assuming $t(0) = 0$. This is seen by considering the trigonometric polynomial

$$t_2 : \theta \mapsto \frac{1}{2}\left\{t(\theta) - t(-\theta)\right\}$$

which belongs to $\mathcal{J}_n$ if t does and is bounded (in absolute value) by 1 if t is. In addition, $t_2(0) = 0$ while $t_2'(0) = t'(0)$. The solution to the problem considered by Bernstein is therefore contained in the following (for references see $\left[15, \text{Chapitre 1}\right]$)

THEOREM A. <u>Let</u> $t \in \mathcal{J}_n$. <u>If</u>

$$(1.1) \qquad \left|t(\theta)\right| \leq 1 \quad \underline{for} \quad \theta \in \mathbb{R}$$

<u>and</u> $t(0) = 0$, <u>then</u>

$$(1.2) \qquad \left|t(\theta)\right| \leq \left|\sin n\theta\right| \quad \underline{for} \quad \theta \in \left[-\frac{\pi}{2n}, \frac{\pi}{2n}\right].$$

<u>The example</u> $t(\theta) := \sin n\theta$ <u>shows that the estimate is sharp</u>

<u>for each</u> θ <u>in</u> $[-\frac{\pi}{2n}, \frac{\pi}{2n}]$.

Dividing the two sides of (1.2) by $|\theta|$ and letting $\theta \to 0$ we obtain $|t'(0)| \leq n$. Hence we have

THEOREM B. <u>If</u> $t \in \mathcal{J}_n$ <u>then</u> (1.1) <u>implies</u>

$$(1.3) \qquad |t'(\theta)| \leq n \quad \underline{for} \quad \theta \in \mathbb{R}.$$

It can be shown that in (1.3) equality holds if and only if $t(\theta) := ae^{in\theta} + be^{-in\theta}$ with $|a| + |b| = 1$.

Theorem B is known as Bernstein's inequality for the derivative of a trigonometric polynomial.

A related but considerably more difficult problem is to estimate $|p'(x_0)|$ at a point $x_0 \in [-1,1]$ if $p(x) := \sum_{\nu=0}^{n} a_\nu x^\nu$ is a polynomial of degree at most n such that

$$(1.4) \qquad |p(x)| \leq 1 \quad for \quad x \in [-1,1].$$

In this case, the answer depends on x_0. Let us take $x_0 = 0$. Considering

$$p_2 : x \mapsto \frac{1}{2}\{p(x) - p(-x)\}$$

we readily see that once again, there is no loss of generality in assuming $p(0) = 0$. The degree of p_2 cannot exceed n-1 if n is even and so <u>in that case</u> (n <u>even</u>) we may not only assume $p(0) = 0$ but may also restrict ourselves to polynomials of degree at most n-1 (rather than n). The problem has thus been reduced to the following:

"Given a polynomial $p(x) := \sum_{\mu=1}^{m} a_\mu x^\mu$ of degree m such that $|p(x)| \leq 1$ for $x \in [-1,1]$ how large can $|a_1|$ be"?

Applying Bernstein's inequality to the trigonometric polynomial $t(\theta) = p(\cos \theta)$ which is of degree m we obtain

$$|(-\sin \theta)p'(\cos \theta)| \leq m \quad for \quad \theta \in \mathbb{R}.$$

The choice $\theta = \frac{\pi}{2}$ gives

$$|a_1| = |p'(0)| \leq m.$$

We therefore have

THEOREM C. If $p(x) := \sum_{\nu=1}^{n} a_\nu x^\nu$ is a polynomial of degree at most n, and $|p(x)| \leq 1$ for $x \in [-1,1]$, then

$$(1.5) \qquad |a_1| \leq \begin{cases} n & \text{if n is odd} \\ n-1 & \text{if n is even.} \end{cases}$$

If we denote by T_m the Chebyshev polynomial of the first kind of degree m and by γ an arbitrary real number then in (1.5) equality holds for $p : x \mapsto e^{i\gamma} T_n(x)$ if n is odd and for $p : x \mapsto e^{i\gamma} T_{n-1}(x)$ if n is even. The following stronger result, analogous to Theorem A, is due to Boas [3, Theorem 3].

THEOREM D. If p is a polynomial of degree at most n, $p(0) = 0$, and $|p(x)| \leq 1$ at the extrema of the Chebyshev polynomial T_m where m = n or n-1 according as n is odd or even, then

$$(1.6) \qquad |p(x)| \leq |T_m(x)| \quad (|x| \leq \sin \frac{\pi}{2m}).$$

There is equality for some x (and hence for all x) if and only if $p(x) \equiv e^{i\gamma} T_m(x)$.

No improvement would result in (1.5) if we assumed $p(x)$ to be real for real x. Recently, it has been proved by Rahman and Ruscheweyh that if p is typically real in $D(0;1) := \{z \in \mathbb{C} : |z| < 1\}$ then $|a_1|$ cannot be larger than 2. Besides p does not have to be a polynomial.

THEOREM E [16, Theorem 4]. If $f(z) := \sum_{\nu=1}^{\infty} a_\nu z^\nu$ is typically real in $D(0;1)$ and $|f(x)| \leq 1$ for $-1 < x < 1$, then

(1.7) $$|a_1| \leq 2.$$

The example $f(z) := \dfrac{2z}{1+z^2}$ shows that (1.7) is sharp.

Why is the bound so much better for typically real functions? Whereas, the Chebyshev polynomial which is extremal in (1.5) has all its zeros on $(-1,1)$, a typically real function has no zeros in $\{z \in \mathbb{C} : 0 < |z| < 1\}$. But this is only part of the reason. In fact, we have

PROPOSITION 1.1. Let $p(z) := z\, q(z)$ where $q(z) := \sum\limits_{\nu=1}^{n} a_\nu z^{\nu-1}$ is a polynomial of degree $n-1$ not vanishing in $D(0;1)$. If $|p(x)| \leq 1$ for $x \in (-1,1)$, then

(1.8) $$|a_1| \leq \sqrt{n}\,\left(1-\tfrac{1}{n}\right)^{-(n-1)/2} \sim \sqrt{en} \quad (\text{as } n \to \infty).$$

We may consider the example

(1.9) $$p(z) := \sqrt{n}\,\left(1-\tfrac{1}{n}\right)^{-(n-1)/2} z(1-z^2)^{(n-1)/2}$$

to see that in (1.8) equality is possible at least for odd n.

In Proposition 1.1 we do not need $p(z)$ to be "real for real z" but the example (1.9) shows that no improvement in the bound for $|a_1|$ would result if this were added as a hypothesis. Thus, requiring a function to be typically real in $D(0;1)$ is much more restrictive than assuming it to be real on $(-1,1)$ and different from zero in $D(0;1) \setminus \{0\}$. This becomes clearer if we replace the hypothesis "$|p(x)| \leq 1$ for $-1 < x < 1$" by "$|p(x)| \leq 1$ for $0 < x < 1$". On the one hand, we have

PROPOSITION 1.2. If $f(z) := \sum\limits_{\nu=1}^{\infty} a_\nu z^\nu$ is typically real in $D(0;1)$ and $|f(x)| \leq 1$ for $0 < x < 1$, then

(1.10) $$|a_1| \leq 4.$$

The example $f(z) := \dfrac{4z}{(1+z)^2}$ shows that (1.10) is sharp.

On the other hand, we have

PROPOSITION 1.3. Let $p(z) := z\, q(z)$ where $q(z) :=$ $\displaystyle\sum_{\nu=1}^{n} a_{\nu}\, z^{\nu-1}$ is a polynomial of degree $n-1$ not vanishing in $D(0;1)$. If $|p(x)| \leq 1$ for $0 < x < 1$, then

(1.11) $$|a_1| \leq n\left(1 - \frac{1}{n}\right)^{-(n-1)} \sim en \quad (\text{as } n \to \infty).$$

The example $p(z) := n\left(1 - \frac{1}{n}\right)^{-(n-1)} z(1-z)^{n-1}$ shows that (1.11) is sharp for each n.

Proposition 1.3 hardly needs a proof. In fact,

$$1 \geq \left|p\left(\tfrac{1}{n}\right)\right| = \tfrac{1}{n}\left|q\left(\tfrac{1}{n}\right)\right| \geq \tfrac{1}{n}\left(1 - \tfrac{1}{n}\right)^{n-1}|a_1|$$

and hence the result.

Instead of proving Proposition 1.1 we shall establish the following more general result.

THEOREM 1.1. Let $f(z) := z\,\dfrac{p(z)}{q(z)}$ where p and q are polynomials of degree μ and ν respectively. If both p and q do not vanish in $D(0;1)$ and $|f(x)| \leq 1$ for $-1 < x < 1$, then

(1.12) $$|f'(0)| \leq \min_{0 < x \leq 1} \frac{1}{x}\,\frac{(1+x^2)^{\nu/2}}{(1-x^2)^{\mu/2}}\,.$$

A straightforward calculation shows that the minimum of $\dfrac{1}{x}\dfrac{(1+x^2)^{\nu/2}}{(1-x^2)^{\mu/2}}$ on the interval $(0,1]$ is attained at the point x_0 where

$$x_0^2 := \begin{cases} \dfrac{\sqrt{(\nu+\mu)^2 + 4(1+\mu-\nu)} - (\nu+\mu)}{2(1+\mu-\nu)} & \text{if } 1+\mu-\nu \neq 0 \\[2mm] \dfrac{1}{\nu+\mu} & \text{if } 1+\mu-\nu = 0. \end{cases}$$

In particular, if $\mu = n-1$ and $\nu = 0$, then $x_0^2 = \frac{1}{n}$ which gives (1.8). In the case $\mu = n-1$, $\nu = n$ which is also of interest we obtain

(1.12') $$|f'(0)| \leq \sqrt{2n}\left(1 - \frac{1}{n}\right)^{-(n-1)/2}.$$

<u>Proof of Theorem 1.1</u>. Using a suggestive notation we may write

$$-f(x) \cdot f(-x) = (f'(0))^2 \, x^2 \prod \left(1 - \frac{x^2}{z_j^2}\right) / \prod \left(1 - \frac{x^2}{\zeta_j^2}\right)$$

where the first product is taken over the zeros of $p(z)$ and the second over those of $q(z)$. The z_j's and the ζ_j's are of modulus ≥ 1. In the case $\mu = 0$, all the z_j's are taken to be ∞. Similarly, all the ζ_j's are taken to be ∞ if $\nu = 0$. Now from the hypothesis "$|f(x)| \leq 1$ for $-1 < x < 1$" we deduce that

$$1 \geq \left| -f(x) \cdot f(-x) \right| \geq \left| f'(0) \right|^2 x^2 \frac{(1-x^2)^\mu}{(1+x^2)^\nu} \quad \text{for } \underline{all} \ x \in (-1,1),$$

i.e.

$$\left| f'(0) \right| \leq \frac{1}{x} \frac{(1+x^2)^{\nu/2}}{(1-x^2)^{\mu/2}} \quad \text{for all } x \in (-1,1).$$

Hence (1.12) must hold.

Proposition 1.2 is a special case of

THEOREM 1.2. <u>Let</u> $f(z) := \sum\limits_{\nu=1}^{\infty} a_\nu z^\nu$ <u>be typically real in</u> $D(0;1)$. <u>If</u> $\left| f(re^{i\alpha}) \right| \leq 1$ <u>for</u> $0 < r < 1$ <u>and some</u> $\alpha \in [0,\pi]$, <u>then</u>

$$(1.13) \quad \left| a_1 \right| \leq \lambda_\alpha := \inf_{0 < r < 1} \frac{1+r}{r(1-r)} \sqrt{1+r^4 - 2r^2 \cos 2\alpha} \ .$$

<u>The estimate is sharp for each</u> α.

Here is a similar generalization of Theorem E.

THEOREM 1.3. <u>Let</u> $f(z) := \sum\limits_{\nu=1}^{\infty} a_\nu z^\nu$ <u>be typically real in</u> $D(0;1)$. <u>If</u> $\left| f(re^{i\alpha}) \right| \leq 1$ <u>for</u> $-1 < r < 1$ <u>and some</u> $\alpha \in [0,\pi)$, <u>then</u>

$$(1.14) \quad \left| a_1 \right| \leq \mu_\alpha := \inf_{0 < r < 1} \frac{1+r^2}{r(1-r^2)} \sqrt{1+r^4 - 2r^2 \cos 2\alpha} \ .$$

<u>The estimate is sharp for each</u> α.

<u>Proof of Theorem 1.2</u>. We recall that $f(z) := \sum\limits_{\nu=1}^{\infty} a_\nu z^\nu$ is typically real in $D(0;1)$ if and only if [17]

$$(1.15) \qquad \text{Re} \left\{ \frac{1-z^2}{z} f(z) \right\} > 0 \quad \text{for } z \in D(0;1).$$

If we set

(1.16)
$$\varphi(z) := \frac{1}{a_1} \frac{1-z^2}{z} f(z)$$

then φ belongs to the class $\mathcal{P}$ of normalized holomorphic functions with positive real part in $D(0;1)$. By a well-known property (see for example [14, p. 40]) of such functions

$$\frac{1}{|a_1|} \frac{|1-r^2 e^{2i\alpha}|}{r} |f(re^{i\alpha})| \geq \frac{1-r}{1+r} \text{ for all } r \in (0,1).$$

Hence

$$|a_1| \leq \frac{1+r}{r(1-r)} |1-r^2 e^{2i\alpha}| \cdot |f(re^{i\alpha})|$$

$$\leq \frac{1+r}{r(1-r)} \sqrt{1+r^4 - 2r^2 \cos 2\alpha} \text{ for all } r \in (0,1)$$

from which the desired result follows.

A straightforward calculation shows that in (1.13) equality holds for the function

$$f_\alpha(z) := \lambda_\alpha \frac{z}{1-z^2} \frac{1-ze^{-i\alpha}}{1+ze^{-i\alpha}}.$$

For the proof of Theorem 1.3 we need the following lemma which is a result of independent interest.

LEMMA 1.1. If $\varphi(z) := 1 + \sum_{\nu=1}^{\infty} c_\nu z^\nu$ belongs to the class $\mathcal{P}$, i.e. φ is holomorphic with Re $\varphi(z) > 0$ in $D(0;1)$, then

(1.17)
$$\max(|\varphi(z)|, |\varphi(-z)|) \geq \frac{1-|z|^2}{1+|z|^2} \text{ for all } z \in D(0;1).$$

Proof. The function

$$\frac{1}{2}\{\varphi(z) + \varphi(-z)\} = 1 + \sum_{\nu=1}^{\infty} c_{2\nu} z^{2\nu}$$

also belongs to $\mathcal{P}$ and so does

$$\psi(z) := 1 + \sum_{\nu=1}^{\infty} c_{2\nu} z^\nu.$$

Hence by the above mentioned property of functions in $\mathcal{P}$

$$|\psi(z)| \geq \frac{1-|z|}{1+|z|} \text{ for all } z \in D(0;1)$$

which implies that

$$\frac{1}{2}|\varphi(z) + \varphi(-z)| = |\psi(z^2)| \geq \frac{1-|z|^2}{1+|z|^2} \text{ for all } z \in D(0;1).$$

This is possible only if (1.17) holds.

<u>Proof of Theorem 1.3</u>. Applying Lemma 1.1 to the function φ defined in (1.16) we obtain

$$\max\left\{\frac{1}{|a_1|}\frac{|1-r^2e^{2i\alpha}|}{r}\left|f(re^{i\alpha})\right|,\ \frac{1}{|a_1|}\frac{|1-r^2e^{2i\alpha}|}{r}\left|f(-re^{i\alpha})\right|\right\}$$

$$\geq \frac{1-r^2}{1+r^2}\ \text{ for all } r\in(0,1).$$

Since both $\left|f(re^{i\alpha})\right|$ and $\left|f(-re^{i\alpha})\right|$ are ≤ 1 for $r\in(0,1)$ we obtain

$$|a_1|\leq\frac{1+r^2}{r(1-r^2)}\left|1-r^2e^{2i\alpha}\right|\ \text{ for all } r\in(0,1).$$

Hence (1.14) holds.

It is easily checked that in (1.14) equality holds for the function

$$f_{(\alpha)}(z):=\mu_\alpha\,\frac{z}{1-z^2}\,\frac{1-z^2e^{-2i\alpha}}{1+z^2e^{-2i\alpha}}\ .$$

REMARKS ON PROPOSITION 1.1. Noting that a_1 is the limiting value of $p(x)/x$ as $x\to 0$ it becomes natural to ask how large $\left|p(x)/x\right|$ can be on $[-1,1]$ if p satisfies the conditions of Proposition 1.1. First, let $0<x\leq c/\sqrt{n}$. Then

$$\left|p(x)p(-x)/x^2\right|\ =\ \left|p'(0)\right|^2\prod_{\nu=1}^{n-1}\left|1-\frac{x^2}{z_\nu^2}\right|$$

$$\leq\ n(1-\tfrac{1}{n})^{-(n-1)}(1+x^2)^{n-1}$$

$$<\ n\,e\,(1+\frac{c^2}{n})^{n-1}$$

$$<\ n\,e^{1+c^2}.$$

Hence for $0<x\leq c/\sqrt{n}$

(1.18) $\quad\min\left\{\left|p(x)/x\right|,\ \left|p(-x)/(-x)\right|\right\}<\sqrt{n}\,e^{(1+c^2)/2}.$

If $c/\sqrt{n}<|x|\leq 1$, then trivially

(1.19) $\quad\quad\quad\quad\left|p(x)/x\right|<\sqrt{n}\,/c.$

From (1.18) and (1.19) we conclude that if c_0 is the (only)

positive root of the equation

(1.20)
$$c \, e^{(1+c^2)/2} = 1,$$

then for $0 < x \leq 1$

(1.21)
$$\min \left\{ \left| p(x)/x \right|, \ \left| p(-x)/(-x) \right| \right\} < \sqrt{n}/c_0.$$

In particular, if the polynomial p happens to be <u>odd</u>, then

$$\left| p(x)/x \right| < \sqrt{n}/c_0 \quad \text{for } \underline{\text{all}} \ x \in \left[-1, 1 \right].$$

Thus we have proved the following

COROLLARY 1.1.   <u>Let</u> $p(z) = z \, q(z)$ <u>be an odd polynomial</u> <u>of degree</u> $\leq n$ <u>such that</u> $\left| p(x) \right| \leq 1$ <u>for</u> $-1 \leq x \leq 1$. <u>If</u> $q(z) \neq 0$ <u>in</u> $D(0;1)$, <u>then for all</u> $x \in \left[-1, 1 \right]$

$$\left| p(x)/x \right| < \sqrt{n}/c_0$$

<u>where</u> c_0 <u>is the only positive root of equation</u> (1.20).

Let p be an <u>odd</u> polynomial of degree n vanishing at the origin and let a be a point of the unit interval such that $\left| p(a) \right| = \max\limits_{-1 \leq x \leq 1} \left| p(x) \right|$. From Theorem D it follows that

(1.22)
$$\left| a \right| \geq \sin \frac{\pi}{2n}.$$

In view of the fact that the extremal polynomial T_n has all its zeros on $(-1,1)$ we may wish to know how small $\left| a \right|$ can be if $p(z) \neq 0$ in $D(0;1)$. It follows readily from Corollary 1.1 that $\left| a \right|$ <u>cannot be smaller than</u> $\dfrac{c_0}{\sqrt{n}}$.

Arguing as in the case of Corollary 1.1   we can deduce from (1.12') the following result.

COROLLARY 1.2.   <u>Let</u> $f(z) := z \, \dfrac{p(z)}{q(z)}$ <u>where</u> p <u>and</u> q <u>are</u> <u>polynomials of degree</u> $\leq n-1$ <u>and</u> $\leq n$ <u>respectively. If both</u> p <u>and</u> q <u>do not vanish in</u> $D(0;1)$ <u>and</u> $\left| f(x) \right| \leq 1$ <u>for</u> $-1 < x < 1$, <u>then</u>

(1.23)
$$\left| f(x)/x \right| \leq \sqrt{2n}/c_1$$

where c_1 <u>is the only positive root of the equation</u>
(1.24)
$$c \, e^{1+2c^2} = 1.$$

A SPECIAL CASE CF THEOREM 1.3. It turns out that the value of μ_α is particularly simple in the case $\alpha = \pi/2$. We have to calculate the minimum of $\Theta(r) := \dfrac{(1+r^2)^2}{r(1-r^2)}$ for $r \in (0,1)$. It is easily checked that the only zero of $\Theta'(r)$ in $(0,1)$ is $\sqrt{2} - 1$ and gives the value 4 as the desired minimum of $\Theta(r)$. In other words $\mu_{\pi/2} = 4$.

1.2. <u>A n   a n a l o g u e   o f   a   t h e o r e m   o f   S c h u r</u>

According to a classical result of Chebyshev if $p(x) := \displaystyle\sum_{\nu=0}^{n} a_\nu x^\nu$ is a polynomial of degree n then
(1.25)
$$|a_n| \leq 2^{n-1} \cdot \max_{-1 \leq x \leq 1} |p(x)|.$$
It is also known [10] that
(1.26)
$$|a_n| \leq \frac{1 \cdot 3 \cdots (2n-1)}{n!} \left(\frac{2n+1}{2}\right)^{1/2} \left(\int_{-1}^{1} |p(x)|^2 \, dx \right)^{1/2}.$$
It was shown by Schur ([20], see Theorem III*) that if $p(1) = 0$, then (1.25) can be replaced by
(1.27)
$$|a_n| \leq 2^{n-1} \left(\cos \frac{\pi}{4n}\right)^{2n} \max_{-1 \leq x \leq 1} |p(x)|.$$
Here we obtain the corresponding improvement in (1.26). In fact, we prove

THEOREM 1.4. <u>If</u> $p(x) := \displaystyle\sum_{\nu=0}^{n} a_\nu x^\nu$ <u>is a polynomial of degree</u> n <u>such that</u> $p(1) = 0$, <u>then</u>
(1.28)
$$|a_n| \leq \frac{n}{n+1} \frac{(2n)!}{2^n (n!)^2} \left(\frac{2n+1}{2}\right)^{1/2} \left(\int_{-1}^{1} |p(x)|^2 \, dx \right)^{1/2}.$$
<u>The inequality is sharp and equality holds for</u>
$$p(x) := P_n(x) - \frac{1}{n^2} \sum_{\nu=0}^{n-1} (2\nu+1) P_\nu(x)$$
<u>where</u> P_ν <u>is the Legendre polynomial of degree</u> ν <u>with the normalization</u> $P_\nu(1) = 1$. <u>Besides,</u>

(1.29) $\quad |a_{n-1}| \leq \frac{(n^2+2)^{1/2}}{n+1} \frac{(2n-2)!}{2^{n-1}((n-1)!)^2} (\frac{2n-1}{2})^{1/2} (\int_{-1}^{1} |p(x)|^2 dx)^{1/2}$

<u>which is again sharp as the example</u>

$$p(x) := \frac{2n+1}{n^2+2} P_n(x) - P_{n-1}(x) + \frac{1}{n^2+2} \sum_{\nu=0}^{n-2} (2\nu+1) P_\nu(x)$$

<u>shows.</u>

In the absence of the hypothesis $p(1) = 0$ the factor $\frac{(n^2+2)^{1/2}}{n+1}$ appearing on the right hand side of (1.29) is to be dropped ($[10]$, see inequality (3)).

<u>Proof of Theorem 1.4.</u> Let

(1.30) $\qquad \varphi_\nu(x) := (\frac{2\nu+1}{2})^{1/2} P_\nu(x)$

$$:= (\frac{2\nu+1}{2})^{1/2} \sum_{j=0}^{[\nu/2]} \frac{(-1)^j (2\nu-2j)!}{2^\nu j!(\nu-j)!(\nu-2j)!} x^{\nu-2j}.$$

Then

$$\int_{-1}^{1} \varphi_\nu(x) \; \varphi_\mu(x) \; dx = \begin{cases} 0 & \text{if } \mu \neq \nu \\ 1 & \text{if } \mu = \nu \end{cases}$$

and the polynomial $p(x)$ can be expressed uniquely in the form

(1.31) $\qquad\qquad\qquad p(x) = \sum_{\nu=0}^{n} \alpha_\nu \cdot \varphi_\nu(x)$

where

$$\sum_{\nu=0}^{n} |\alpha_\nu|^2 = \int_{-1}^{1} |p(x)|^2 dx .$$

From (1.31) in conjunction with (1.30) it follows that

(1.32) $\quad a_n = (\frac{2n+1}{2})^{1/2} \frac{(2n)!}{2^n (n!)^2} \alpha_n \; , \; a_{n-1} = (\frac{2n-1}{2})^{1/2} \frac{(2n-2)!}{2^{n-1}((n-1)!)^2} \alpha_{n-}$

Now we wish to prove that if $\gamma_\mu > \gamma_\nu \geq 0$ for $\nu = 0, 1, \ldots,$ $\mu-1, \mu+1, \ldots, n$ then under the hypothesis of the theorem

(1.33) $\qquad\qquad \sum_{\nu=0}^{n} \gamma_\nu |\alpha_\nu|^2 \leq (\gamma_\mu - \gamma) \sum_{\nu=0}^{n} |\alpha_\nu|^2$

where γ is the unique root of the equation

(1.34) $\qquad\qquad \sum_{\nu=0}^{n} \frac{2\nu+1}{\gamma_\mu - \gamma_\nu - x} = 0$

in $(0, \Gamma := \min_{0 \leq \nu \leq n; \nu \neq \mu} (\gamma_\mu - \gamma_\nu))$.

We write the left-hand side of (1.33) as

$$\sum_{\nu=0}^{n} \gamma_\nu |\alpha_\nu|^2 = \gamma_\mu \sum_{\nu=0}^{n} |\alpha_\nu|^2 - \sum_{\nu=0;\nu\neq\mu}^{n} (\gamma_\mu - \gamma_\nu)|\alpha_\nu|^2$$

$$= \gamma_\mu \sum_{\nu=0}^{n} |\alpha_\nu|^2 - \sum_{\nu=0;\nu\neq\mu}^{n} (\gamma_\mu - \gamma_\nu - \gamma)|\alpha_\nu|^2 - \gamma \sum_{\nu=0;\nu\neq\mu}^{n} |\alpha_\nu|^2$$

where, for the moment, γ is a constant in $(0, \Gamma)$. From the hypothesis $p(1) = 0$ and Schwarz's inequality we obtain

$$\left| \left(\frac{2\mu+1}{2}\right)^{1/2} \alpha_\mu \right|^2 = \left| \sum_{\nu=0;\nu\neq\mu}^{n} \left(\frac{2\nu+1}{2}\right)^{1/2} \alpha_\nu \right|^2$$

$$\leq \left\{ \sum_{\nu=0;\nu\neq\mu}^{n} \left(\frac{2\nu+1}{2}\right)^{1/2} |\alpha_\nu| \right\}^2$$

$$= \left\{ \sum_{\nu=0;\nu\neq\mu}^{n} (\gamma_\mu - \gamma_\nu - \gamma)^{1/2} |\alpha_\nu| \cdot \left(\frac{2\nu+1}{2}\right)^{1/2} (\gamma_\mu - \gamma_\nu - \gamma)^{-1/2} \right\}^2$$

$$\leq \sum_{\nu=0;\nu\neq\mu}^{n} (\gamma_\mu - \gamma_\nu - \gamma)|\alpha_\nu|^2 \cdot \sum_{\nu=0;\nu\neq\mu}^{n} \frac{2\nu+1}{2} (\gamma_\mu - \gamma_\nu - \gamma)^{-1},$$

so that

$$-\sum_{\nu=0;\nu\neq\mu}^{n} (\gamma_\mu - \gamma_\nu - \gamma)|\alpha_\nu|^2 \leq -\frac{2\mu+1}{2}|\alpha_\mu|^2 \left\{ \sum_{\nu=0;\nu\neq\mu}^{n} \frac{2\nu+1}{2} (\gamma_\mu - \gamma_\nu - \gamma)^{-1} \right\}^{-1}.$$

Now if γ happens to be the root of the equation (1.34) lying in $(0, \Gamma)$, then

$$\left\{ \sum_{\nu=0;\nu\neq\mu}^{n} \frac{2\nu+1}{2} (\gamma_\mu - \gamma_\nu - \gamma)^{-1} \right\}^{-1} = \frac{2}{2\mu+1} \gamma$$

and we get

$$\sum_{\nu=0}^{n} \gamma_\nu |\alpha_\nu|^2 \leq \gamma_\mu \sum_{\nu=0}^{n} |\alpha_\nu|^2 - \gamma|\alpha_\mu|^2 - \gamma \sum_{\nu=0;\nu\neq\mu}^{n} |\alpha_\nu|^2 = (\gamma_\mu - \gamma) \sum_{\nu=0}^{n} |\alpha_\nu|^2$$

which proves (1.33).

If $\gamma_n = 1$, $\gamma_\nu = 0$ for $\nu = 0, 1, \ldots, n-1$ then γ turns out to be equal to $\frac{2n+1}{(n+1)^2}$ and (1.33) reduces to

$$(1.35) \qquad |\alpha_n| \leq \frac{n}{n+1} \left(\sum_{\nu=0}^{n} |\alpha_\nu|^2 \right)^{1/2}.$$

Similarly, choosing $\gamma_{n-1} = 1$, $\gamma_\nu = 0$ for $\nu = 0, 1, \ldots, n-2, n$ we obtain

(1.36)
$$\left|\alpha_{n-1}\right| \leq \frac{(n^2+2)^{1/2}}{n+1} \left(\sum_{\nu=0}^{n} \left|\alpha_\nu\right|^2\right)^{1/2}.$$

Combining (1.35), (1.36) with (1.32) we readily obtain (1.28), (1.29) respectively.

Both the inequalities (1.28), (1.29) are sharp and in each case the extremal polynomials are easily identified.

1.3. Inequalities for functions of exponential type

Let $f(z)$ be an entire function and for each $r>0$ let
$$M(r) := \max_{|z|=r} \left|f(z)\right|.$$
The function $f(z)$ is said to be of order ρ if
$$\limsup_{r \to \infty} \frac{\log \log M(r)}{\log r} = \rho.$$
In the case $0 < \rho < \infty$, the function $f(z)$ is said to be of type T if
$$\limsup_{r \to \infty} \frac{\log M(r)}{r^\rho} = T.$$
An entire function is said to be of exponential type τ ($<\infty$) if it is either of order <1 or of type <u>at most</u> τ <u>if of order</u> 1, i.e. for each $\varepsilon > 0$
$$\left|f(z)\right| = O\left(e^{(\tau+\varepsilon)|z|}\right) \quad \text{for all } z \in \mathbb{C}.$$
One of the most interesting examples of an entire function of exponential type is a trigonometric polynomial. In fact, $t(x) := \sum_{\nu=-n}^{n} a_\nu e^{i\nu x}$ is the restriction to the real axis of the entire function $t(z)$ which is of exponential type n.

Theorem B was extended to entire functions of exponential type by S. Bernstein who proved:

THEOREM B'. <u>Let</u> $f(z)$ <u>be an entire function of exponential</u>

type τ <u>such that</u> $|f(x)| \leq 1$ <u>for</u> $-\infty < x < \infty$. <u>Then</u>

(1.37) $|f'(x)| \leq \tau$ <u>for</u> $-\infty < x < \infty$.

The examples $e^{i\tau z}$, $\sin \tau z$ and $\cos \tau z$ show that (1.37) is sharp.

Using an approximation method due to Lewitan [11], in a form given by Hörmander [9], Frappier [6, Theorem 5] has recently obtained the following generalization of Theorem A.

THEOREM A'. <u>Let</u> $f(z)$ <u>be an entire function of exponential type</u> τ <u>such that</u> $|f(x)| \leq 1$ <u>for</u> $x \in \mathbb{R}$ <u>and</u> $f(0) = 0$. <u>Then</u>

(1.38) $|f(x)| \leq |\sin \tau x|$ <u>for</u> $|x| \leq \frac{\pi}{2\tau}$.

We observe that Theorem A' is implicitly contained in a result stated in the above mentioned paper of Hörmander. According to that result (see the remark following the Corollary on page 26 Of [9]) we have

LEMMA 1.2. <u>Let</u> $g(z)$ <u>be an entire function of exponential type</u> τ <u>such that</u> $g(x)$ <u>is real and</u> $-1 \leq g(x) \leq 1$ <u>when x is real.</u> <u>If</u> $g(0) = \cos a$, <u>where</u> $0 \leq a < \pi$ <u>and</u> $g'(0) = 0$, <u>then</u>

$$|g(x)| \geq \cos(\tau^2 x^2 + a^2)^{1/2} \text{ when } \tau^2 x^2 + a^2 \leq \pi^2.$$

Now let $f(z)$ satisfy the conditions of Theorem A' and consider the function

$$F(z) = 1 - f(z)\overline{f(\bar{z})}$$

which is of exponential type 2τ with $F(0) = 1$, $F'(0) = 0$. Since $F(x) \geq 0$ for $x \in \mathbb{R}$ we may write (see $\left[1, \text{ p. } 154\right]$ or $\left[2, \text{ Section } 7.5\right]$)

$$F(x) = |\varphi(x)|^2$$

where φ is an entire function of exponential type τ such that $\varphi(0) = 1$, $\varphi'(0) = 0$ and $|\varphi(x)| \leq 1$ for $x \in \mathbb{R}$. Thus

Lemma 1.2 applies with a = 0 to the function

g(z) = (φ(z) + $\overline{φ(\bar{z})}$)/2 and we obtain

$$|φ(x)| \geq g(x) \geq \cos τx \quad \text{for} \quad |x| \leq \frac{π}{τ}.$$

Consequently,

(1.39) $1 - |f(x)|^2 = F(x) \geq \cos^2 τx \quad \text{for} \quad |x| \leq \frac{π}{2τ}$

which is equivalent to (1.38). In (1.39) we have used the fact

that $\cos τx \geq 0$ for $|x| \leq \frac{π}{2τ}$.

Note that in our proof of Theorem A' we have not required

f(x) to be real for real x.

From Lemma 1.2 we also deduce the following result whose

relevance is abundantly clear.

THEOREM 1.5. <u>Let</u> f(z) <u>be an entire function of exponential</u>

<u>type</u> τ <u>such that</u> $|f(x)| \leq 1$ <u>for</u> $x \in \mathbb{R}$. <u>If</u> $|f(0)| = \cos a$, <u>where</u>

$0 \leq a \leq \frac{π}{2}$, <u>and</u> f'(0) = 0, <u>then</u>

(1.40) $|f(x)| \leq \sin(\sqrt{(π-a)^2 + τ^2 x^2} - \frac{π}{2})$ <u>for</u> $|x| \leq \frac{\sqrt{a(2π-a)}}{τ}$.

<u>Proof of Theorem 1.5.</u> First note that in Lemma 1.2, the

hypothesis "g(0) = cos a" may be replaced by "g(0) $\geq$ cos a"

without any change in the conclusion. In order to prove that

(1.40) holds at an arbitrary point $x_0 \in \left[-\frac{\sqrt{a(2π-a)}}{τ} , \frac{\sqrt{a(2π-a)}}{τ} \right]$

we may clearly assume $f(x_0) \neq 0$. Now choose $γ \in \mathbb{R}$ such that

$f(x_0)e^{iγ}$ is positive. The function

$$g(z) := -\frac{1}{2} \left\{ f(z)e^{iγ} + \overline{f(\bar{z})}e^{-iγ} \right\}$$

is entire and of exponential type τ. It is real and

$-1 \leq g(x) \leq 1$ when x is real. Further, $g(0) \geq - \cos a = \cos(π-a)$

and g'(0) = 0. Hence

$$g(x) \geq \cos(τ^2 x^2 + (π-a)^2)^{1/2} \quad \text{for} \quad |x| \leq \frac{\sqrt{a(2π-a)}}{τ}.$$

Since $g(x_0) = -f(x_0)e^{i\gamma}$ we obtain

$$\left|f(x_0)\right| = f(x_0)e^{i\gamma} \leq -\cos(\tau^2 x_0^2 + (\pi-a)^2)^{1/2} = \sin(\sqrt{(\pi-a)^2 + \tau^2 x_0^2} - \tfrac{\pi}{2}).$$

From Lemma 1.2 and Theorem 1.5 we can easily deduce the following

COROLLARY 1.3. <u>Under the conditions of Theorem 1.5 we</u> <u>have</u>

$$(1.41) \qquad\qquad \left|f''(0)\right| \leq \frac{\sin a}{a}\,\tau^2 \ .$$

Inspite of its simplicity, the result contained in this corollary does not seem to have been noted before.

CHAPTER 2

ON THE LOCATION CF THE CRITICAL PCINTS CF A PCLYN

We shall denote by $D(a;R)$ the open disk $\left\{ z \in \mathbb{C} : |z-a| < R \right\}$ and by $\overline{D(a;R)}$ its closure. According to a well-known theorem of Gauss every convex domain containing all the zeros of a polynomial $p(z) := c \prod_{\nu=1}^{n} (z-z_\nu)$ also contains all the zeros of the derivative $p'(z)$. In particular, if all the zeros $z_1, \ldots, z_n$ of $p(z)$ lie in $\overline{D(0;1)}$ then so do all the zeros of $p'(z)$. It was conjectured by Bl. Sendov that, for $n \geq 2$, $p'(z)$ has at least one zero in each of the disks $\overline{D(z_\nu;1)}$, $\nu=1, \ldots,$ n; the example $p(z) := z^n - 1$ shows that the open disks $D(z_\nu;1)$, $\nu = 1, \ldots,$ n may not contain any zero of $p'(z)$. After the conjecture was published in $\left[8, \text{ Problem } 4.5 \right]$ it attracted the attention of many mathematicians. It has been verified in various special cases, e.g.

1. It is true $\left[18, \text{ Satz } 3 \right]$ if $p(0) = 0$.

2. For each zero z_ν with $|z_\nu| = 1$ the disk $\overline{D(z_\nu;1)}$ does contain a zero of $p'(z)$ $\left(\left[7, 18, 13 \right] \right)$.

3. The conjecture holds for polynomials of degree $n \leq 5$ $\left[13 \right]$. For a historical background and the progress made towards the solution of the problem we refer the reader to $\left[12 \right]$, $\left[19 \right]$ and $\left[4 \right]$. According to a recent result of Bojanov, Rahman and Szynal $\left[5 \right]$ if $p(z) := c \prod_{\nu=1}^{n} (z-z_\nu)$ is a polynomial of degree $n \geq 2$ with $|z_\nu| \leq 1$ for $\nu = 1, \ldots,$ n then each of the disks $D(z_\nu; 2^{1/n})$ contains at least one zero of $p'(z)$.

Since $2^{1/n} < 1 + \frac{1}{n}$ for $n \geq 2$ and decreases to 1 as $n \to \infty$ we may say that Sendov's conjecture is "asymptotically" true. However, it does remain to be decided whether the number $2^{1/n}$ appearing in the result of Bojanov, Rahman and Szynal can be replaced by 1 for $n \geq 6$ as it can be done for $n \leq 5$.

The disk $\overline{D(z_\nu;1)}$ trivially contains a zero of $p'(z)$ if z_ν happens to be a multiple zero of $p(z)$. A non-trivial question in the case of a zero a of $p(z)$ of multiplicity k is to ask for the smallest number R_k such that the disk $\overline{D(a;R_k)}$ contains at least k zeros of $p'(z)$. In the present chapter we attempt to answer this (more general) question. Here are our main results.

THEOREM 2.1. Let $|a| = 1$. If $p(z) := c(z-a)^k \prod_{\nu=1}^{n-k} (z-z_\nu)$ is a polynomial of degree n ($> k$) such that $|z_\nu| \leq 1$ for $\nu = 1, \ldots, n-k$, then taking multiplicity into account, $p'(z)$ has at least k zeros in $\overline{D(\frac{a}{k+1};\frac{k}{k+1})}$ $(\subset \overline{D(a;\frac{2k}{k+1})})$.

THEOREM 2.2. Let $|a| \leq 1$. If $p(z) := c(z-a)^k \prod_{\nu=1}^{n-k} (z-z_\nu)$ is a polynomial of degree n with $k < n \leq (k+1)^2$ such that $|z_\nu| \leq 1$ for $\nu = 1, \ldots, n-k$, then taking multiplicity into account, $p'(z)$ has at least k zeros in $D(a;\frac{2k}{k+1})$.

Before presenting the proofs we wish to discuss the sharpness of our results. Let α_k be the unique number in $(\pi/2,\pi)$ such that

$$\cos \alpha_k = - \frac{(k+1)^2-2}{(k+1)^2}$$

and consider the particular polynomial

$$p(z) := (z-1)^k (z^2 - 2z \cos \alpha + 1).$$

Then, in addition to a $(k-1)$-fold zero at 1, $p'(z)$ has zeros

at
$$w_1, \ w_2 := \frac{1 + (k+1)\cos\alpha \pm i\sqrt{(k+2)(k+2\cos\alpha) - \{1 + (k+1)\cos\alpha\}^2}}{k+2}$$

where the quantity under the radical sign is positive if $0 < \alpha < \alpha_k$. It is easily checked that as α runs from 0 to α_k the points w_1 and w_2 describe the boundary of $\overline{D(\frac{1}{k+1}; \frac{k}{k+1})}$. This proves the sharpness of Theorem 2.1. In order to see that Theorem 2.2 is also the best possible result of its kind let us take $p(z) := (z+1)(z-1)^k$. Then $p'(z)$ has a $(k-1)$-fold zero at 1 and a simple zero at $-\frac{k-1}{k+1}$ which shows that the open disk $D(a; \frac{2k}{k+1})$ may not contain more than $k-1$ zeros of $p'(z)$. As another example we may consider

$$p(z) := (z^2 + 2\frac{(k+1)^2 - 2}{(k+1)^2} z + 1)(z-1)^k$$

whose derivative has a double zero at $-\frac{k-1}{k+1}$ in addition to a $(k-1)$-fold zero at 1.

Proof of Theorem 2.1. Let us write

(2.1) $$p(z) = (z-a)^k q(z).$$

Without loss of generality we may assume $q(a) \neq 0$. Let us denote by $w_1, \ \ldots, \ w_{n-k}$ the zeros of $p'(z)$ other than the $(k-1)$-fold zero at a. Since

$$\frac{p''(z)}{p'(z)} = \frac{k-1}{z-a} + \sum_{j=1}^{n-k} \frac{1}{z - w_j}$$

the function

(2.2) $$f(z) := \sum_{j=1}^{n-k} \frac{1}{z - w_j} = \frac{p''(z)}{p'(z)} - \frac{k-1}{z-a} = \frac{(z-a)p''(z) - (k-1)p'(z)}{(z-a)p'(z)}$$

is holomorphic in a neighbourhood of the point a. From (2.1) we have

$$p'(z) = (z-a)^k q'(z) + k(z-a)^{k-1} q(z),$$
$$p''(z) = (z-a)^k q''(z) + 2k(z-a)^{k-1} q'(z) + k(k-1)(z-a)^{k-2} q(z)$$

so that

$$f(z) = \frac{(z-a)q''(z) + (k+1)q'(z)}{(z-a)q'(z) + k\,q(z)} \,.$$

In particular, we obtain

(2.3) $\qquad \displaystyle\sum_{j=1}^{n-k} \frac{1}{a - w_j} = f(a) = \frac{k+1}{k} \frac{q'(a)}{q(a)} \,.$

Since $q(z) := c \displaystyle\prod_{j=1}^{n-k} (z - z_j)$, we have

$$\frac{q'(a)}{q(a)} = \sum_{j=1}^{n-k} \frac{1}{a - z_j}$$

and so

(2.4) $\qquad \displaystyle\sum_{j=1}^{n-k} \frac{a}{a - w_j} = \frac{k+1}{k} \sum_{j=1}^{n-k} \frac{a}{a - z_j}.$

Taking real parts of the two sides of (2.4), we get

(2.5) $\qquad \displaystyle\sum_{j=1}^{n-k} \operatorname{Re} \frac{a}{a - w_j} = \frac{k+1}{k} \sum_{j=1}^{n-k} \operatorname{Re} \frac{a}{a - z_j}.$

The hypothesis $|z_j| \le 1$ implies that $\operatorname{Re} \dfrac{a}{a - z_j} \ge \dfrac{1}{2}$ for $1 \le j \le n-k$.

Thus

$$\sum_{j=1}^{n-k} \operatorname{Re} \frac{a}{a - w_j} = \frac{k+1}{k} \frac{n-k}{2}$$

and $\operatorname{Re} \dfrac{a}{a - w_j} \ge \dfrac{k+1}{2k}$ for some j ($1 \le j \le n-k$). This proves that

$p'(z)$ has at least one zero $\ne a$ in $\overline{D(\frac{a}{k+1}; \frac{k}{k+1})}$. Since it has a

$(k-1)$-fold zero at a the theorem follows.

$\qquad$ Proof of Theorem 2.2. According to Theorem 2.1, the disk

$\overline{D(a; \frac{2k}{k+1})}$ contains at least k zeros of $p'(z)$ in the case $|a| = 1$

if only $n > k$. So we may assume $0 \le |a| < 1$. Besides, there is no

loss of generality in supposing that $0 \le a < 1$. Then clearly the

polynomial

$$P(z) := (az - 1)^n \, p(\tfrac{z - a}{az - 1})$$

has a zero of multiplicity k at the origin and all its other

zeros lie in $\overline{D(0;1)}$. Hence it has the form

$P(z) = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_k z^k = z^k (c_n z^{n-k} + c_{n-1} z^{n-k-1} + \cdots + c$

where $|c_k| \le |c_n|$ and $|c_{n-1}| \le (n-k)|c_n|$. Since

$$P'(z) = na \ (az-1)^{n-1} \ p(\tfrac{z-a}{az-1}) - (1-a^2)(az-1)^{n-2} \ p'(\tfrac{z-a}{az-1})$$

we have

$$(1-a^2)(az-1)^{n-1} \ p'(\tfrac{z-a}{az-1}) = na \ P(z) - (az-1)P'(z)$$

$$= (nc_n + ac_{n-1})z^{n-1} + \cdots + kc_k z^{k-1} \ .$$

Consequently, the zeros of $p'(\tfrac{z-a}{az-1})$ are the same as those of the polynomial

$$z^{k-1} \ (z^{n-k} + \cdots + \frac{kc_k}{nc_n + ac_{n-1}}),$$

i.e. it has a $(k-1)$-fold zero at the origin and at least one

zero in $\left\{z: 0 < |z| \leq (\frac{k}{n-(n-k)a})^{1/(n-k)}\right\}$. Hence $p'(\tfrac{z-a}{az-1})$ has

at least k zeros in $D(0;(\frac{k}{n-(n-k)a})^{1/(n-k)})$. Setting

$$\eta := (\frac{k}{n-(n-k)a})^{1/(n-k)}$$

we readily deduce that $p'(z)$ has at least k zeros in

$D(\frac{a(1-\eta^2)}{1-a^2\eta^2};\frac{\eta(1-a^2)}{1-a^2\eta^2})$. It is now enough to show that

(2.6) $$D(\frac{a(1-\eta^2)}{1-a^2\eta^2};\frac{\eta(1-a^2)}{1-a^2\eta^2}) \subseteq D(a;\frac{2k}{k+1}).$$

Since the disk on the left-hand side of (2.6) has the real

interval $\left[\frac{a-\eta}{1-a\eta},\frac{a+\eta}{1+a\eta}\right]$ as a diameter it is sufficient to check

that

$$a - \frac{a-\eta}{1-a\eta} \leq \frac{2k}{k+1}$$

which is equivalent to

(2.7) $$(1-a^2) \frac{k+1}{2k} + a \leq (\frac{n-(n-k)a}{k})^{1/(n-k)} .$$

Setting $n = \lambda k$ in (2.7) we see that the theorem will be

proved if we show that the inequality

(2.8) $$f(k,a,\lambda) := (\lambda-1)a + \left\{a + \frac{k+1}{2k}(1-a^2)\right\}^{(\lambda-1)k} \leq \lambda$$

holds for $0 \leq a \leq 1$ and $1 < \lambda \leq \frac{(k+1)^2}{k}$. Since f is a continuous

function of a and assumes the value λ for a = 1 it is more

than enough to demonstrate that it increases with a on the interval $[0,1)$. Calculating the partial derivative of f with respect to a we obtain

$$\frac{1}{\lambda-1}\frac{\partial}{\partial a} f(k,a,\lambda) = 1 + g(k,a,\lambda)$$

where

$$g(k,a,\lambda) := \left\{k-(k+1)a\right\}\left\{a + \frac{k+1}{2k}(1-a^2)\right\}^{(\lambda-1)k-1}.$$

Since $g(k,a,\lambda)$ is positive for $0 \le a < \frac{k}{k+1}$ the same can be said about $\frac{\partial}{\partial a} f(k,a,\lambda)$. Now we show that $g(k,a,\lambda)$ decreases from $g(k,\frac{k}{k+1},\lambda) = 0$ to $g(k,1,\lambda) = -1$ as a increases from $\frac{k}{k+1}$ to 1. For this we have to look at the sign of $\frac{\partial}{\partial a} g(k,a,\lambda)$ for $a \in \left[\frac{k}{k+1},1\right)$. A simple calculation shows that

$$\frac{\partial}{\partial a} g(k,a,\lambda) < 0$$

if and only if

$$(2.9) \quad \left\{(\lambda-1)k-1\right\}\left\{k-(k+1)a\right\}^2 < \frac{k+1}{2}\left\{2ak + (k+1)(1-a^2)\right\}.$$

We will like this to be true for $1 < \lambda \le \frac{(k+1)^2}{k}$. Since the left-hand side of (2.9) is an increasing function of λ whereas the right-hand side does not depend on it we need to verify the inequality only for $\lambda = \frac{(k+1)^2}{k}$. In that case it reduces to

$$(k^2+k)\left\{k-(k+1)a\right\}^2 < \frac{k+1}{2}\left\{2ak + (k+1)(1-a^2)\right\}$$

which is equivalent to

$$(2.10) \qquad\qquad (a-\frac{k-1}{k+1})(a-1) < 0.$$

This latter inequality is obviously true for $a \in (\frac{k-1}{k+1},1)$ and so a fortiori for $a \in \left[\frac{k}{k+1},1\right)$. Thus (2.9) holds for $\lambda = \frac{(k+1)^2}{k}$ and consequently for $\lambda < \frac{(k+1)^2}{k}$ as well. This implies that

$$g(k,a,\lambda) > g(k,1,\lambda) = -1 \text{ for } \frac{k}{k+1} \le a < 1.$$
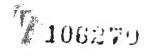
Hence $\frac{\partial}{\partial a} f(k,a,\lambda) = (\lambda-1)\left\{1 + g(k,a,\lambda)\right\}$ is positive not only for $0\leq a<\frac{k}{k+1}$ but also for $\frac{k}{k+1}\leq a<1$, i.e. $f(k,a,\lambda)$ increases from $f(k,0,\lambda)$ to $f(k,1,\lambda) = \lambda$ as a increases from 0 to 1. With this the proof of inequality (2.8) is complete and so is the proof of Theorem 2.2.

Theorem 2.2 implies in particular that if $|a|\leq 1$ and $p(z):= c(z-a)^k \prod_{j=1}^{n-k}(z-z_j)$ is a polynomial of degree n with $2\leq k<n\leq k+4$ and $|z_j|\leq 1$ for $j = 1, \ldots, n-k$, then $p'(z)$ has at least k zeros in $D(a;\frac{2k}{k+1})$. Since this is also known to be true $\begin{bmatrix}13\end{bmatrix}$ for $k = 1$ we may state the following

COROLLARY 2.1. Let $|a|\leq 1$. If $p(z):= c(z-a)^k \prod_{j=1}^{n-k}(z-z_j)$ is a polynomial of degree n with $k<n\leq k+4$ such that $|z_j|\leq 1$ for $j = 1, \ldots, n-k$, then taking multiplicity into account, $p'(z)$ has at least k zeros in $D(a;\frac{2k}{k+1})$.

R E F E R E N C E S

1. N. I. Achieser, <u>Theory of Approximation</u>, Frederick Ungar
   Publishing Co., New York, 1956.

2. R. P. Boas, Jr., <u>Entire Functions</u>, Academic Press, New
   York, 1954.

3. ——————————, <u>Inequalities for polynomials with a
   prescribed zero</u>, Studies in mathematical analysis
   and related topics (Essays in honour of George Póly
   Stanford University Press, Stanford, California,
   1962, 42-47.

4. B. D. Bojanov, Q. I. Rahman and J. Szynal, <u>On a conjectur
   about the critical points of a polynomial</u>, Delay
   Equations, Approximation and Application (Interna-
   tional Symposium, University of Mannheim, October
   8-11, 1984), pp. 83-93, <u>Internat. Schriftenreihe
   Numer. Math</u>., 74, Birkhäuser Verlag, Basel Boston
   Stuttgart, 1985.

5. ——————————————————, <u>On a conjectur
   of Sendov about the critical points of a polynomia]</u>
   Math. Z. 190 (1985), 281-285.

6. C. Frappier, <u>Inequalities for entire functions of expon-
   ential type</u>, Canad. Math. Bull. 27 (1984), 463-471.

7. A. W. Goodman, Q. I. Rahman and J. S. Ratti, <u>On the zeros
   of a polynomial and its derivative</u>, Proc. Amer.
   Math. Soc. 21 (1969), 273-274.

8. W. K. Hayman, <u>Research Problems in Function Theory</u>, The
   Athlone Press of the University of London, London,
   1967.

9. L. Hörmander, <u>Some inequalities for functions of expone-
   ntial type</u>, Math. Scand. 3 (1955), 21-27.

10. G. Labelle, <u>Concerning polynomials on the unit interval</u>,
    Proc. Amer. Math. Soc. 20 (1969), 321-326.

11. B. M. Lewitan, Über eine Verallgemeinerung der Ungleichungen von S. Bernstein und H. Bohr, Doklady Akad. Nauk SSSR 15 (1937), 169-172.

12. M. Marden, Conjectures on the critical points of a polynomial, Am. Math. Monthly 90 (1983), 267-276.

13. A. Meir and A. Sharma, On Ilyeff's conjecture, Pac. J. Math. 31 (1969), 459-467.

14. Ch. Pommerenke, Univalent Functions (with a chapter on quadratic differentials by G. Jensen), Vandenhoeck and Ruprecht, Götingen, 1975.

15. Q. I. Rahman and G. Schmeisser, Les inégalités de Markoff et de Bernstein, Séminaire de Mathématiques Supérieures, No. 86 (Eté, 1981). Les Presses de l'Université de Montréal, Qué. 1983, 173 pp.

16. Q. I. Rahman and St. Ruscheweyh, Markov's inequality for typically real polynomials (to appear).

17. W. W. Rogosinski, Über positive harmonische Sinusentwicklungen, Jber. Deutsch. Math.-Verein. 40 (1931), 33-35.

18. G. Schmeisser, Bemerkungen zu einer Vermutung von Ilieff, Math. Z. 111 (1969), 121-125.

19. —————, On Ilieff's conjecture, Math. Z. 156 (1977), 165-173.

20. I. Schur, Über das Maximum des absoluten Betrages eines Polynoms in einem gegebenen Intervall, Math. Z. 4 (1919), 271-287.

MATH-1986-D-TAR-CON